ABSTRACT

A method and apparatus for designing low-order linear-phase IIR filters is disclosed. Given an FIR filter, the method utilizes a new Krylov subspace projection method, called the rational Arnoldi method with adaptive orders, to synthesize an approximated IIR filter with small orders. The method is efficient in terms of computational complexity. The synthesized IIR filter can truly reflect essential dynamical features of the original FIR filter and indeed satisfies the design specifications. In particular, the linear-phase property is retained in the passband.

1 Claim, 13 Drawing Sheets
Start

Input the design specifications of a digital filter

Design an FIR filter satisfying the design specifications

Design low-order linear-phase IIR filters using the model-order reduction techniques

End
Start 20

Step 1
Input the design specifications of a digital filter

Step 2
Design an FIR filter satisfying the design specifications and store the impulse responses

Step 3
Construct the state space matrices \( \{A, b, c\} \)

Step 4
Generate the orthogonal projection matrix \( V \) using the rational Arnoldi method with adaptive orders

Step 5
Generate low-order FIR filters approximating the original FIR filter

End 32

FIG.2
Start

Step 1

Compute the first Krylov vector for each expansion point

Step 2

Set \( j = 1 \)

Step 3

Set \( z_i \) be the expansion point giving the greatest moment difference

Step 4

Generate the corresponding orthogonal vector

Step 5

Update the new residual vector about each expansion point for the next iteration

Step 6

Set \( j = j + 1 \)

Step 7

\( j \leq q \)?

Step 8

Generate the orthogonal projection matrix

End

FIG. 4
FIG. 5C
FIG. 6B

Error in magnitude

$\omega/n (\text{rad/s})$
FIG. 6C
FIG. 7B
FIG. 7C
EFFICIENT DIGITAL FILTER DESIGN TOOL FOR APPROXIMATING AN FIR FILTER WITH A LOW-ORDER LINEAR-PHASE IIR FILTER

BACKGROUND OF THE INVENTION

1. Field of the Invention

The present invention relates to a method of designing digital filters, and more particularly, to a method of designing low-order linear-phase IIR filters for approximating an FIR filter.

2. Description of Related Art

Digital filters are used in many signal processing applications. One type of digital filter is an impulse response filter (IRF) filter. The advantages of the IRF filters are that they are linear phase and can be designed to have a specific frequency response. However, when the specification is one that requires a large number of coefficients, the resulting filter may not have the desired characteristics.

There are two classes of digital filters. These are finite duration impulse response (FIR) filters and infinite duration impulse response (IIR) filters. The FIR filters are exact linear phase and guaranteed stable. However, when the specification is one that requires a large number of coefficients, the resulting filter may not have the desired characteristics.

One way to synthesize IIR filters with linear phase in the passband is to solve the rational approximation problem using the various methods. These methods, for example, include the pole approximation, linear programming, nonlinear programming, multiple criterion optimization and eigenfilter approach.

Another way is called the indirect approach. It will be composed of three steps, as shown in FIG. 1. Step 1 receives and stores the design specifications of a digital filter. These specifications are required in the frequency domain in terms of the desired magnitude and phase response of the filter. Then, a linear-phase FIR filter, which meets design specifications, will be designed in step 2. The order and the coefficients of the IIR filters can be obtained using the conventional methods such as the frequency sampling design technique, the window design technique and the optimal equiripple design technique. Finally, a lower-order IIR filter will be obtained using filter approximation techniques in step 3.

FIG. 1 illustrates the design flow of the low-order linear-phase IIR filter.

FIG. 2 illustrates the design flow of the low-order linear-phase IIR filter.

BRIEF DESCRIPTION OF THE DRAWINGS

FIG. 1 illustrates the design flow of the low-order linear-phase IIR filter.

FIG. 2 illustrates the design flow of the low-order linear-phase IIR filter.

FIG. 3 illustrates the typical design specifications of a low-pass filter.

FIG. 4 shows the detail flow of the rational Arnoldi method with adaptive orders.

FIGS. 5A-7C show the bode plots of the magnitude, the error in magnitude, and the phase of the original FIR filter and the low-order IIR filter.

DETAILED DESCRIPTION OF THE PREFERRED EMBODIMENT

FIG. 2 shows the design flow of the low-order linear-phase IIR filter in the present invention in which includes the steps of receiving and storing the design specifications in step 1, designing an FIR filter satisfying the design specifications and saving the order and coefficients in step 2, establishing the state space matrices \( \{A, b, c\} \) in step 3, performing the rational Arnoldi method with adaptive orders and producing the orthogonal projection matrix \( V \) in step 4, and generating the corresponding low-order linear-phase IIR filter, which can approximate the original FIR filter and satisfy the design specifications in step 5.

FIG. 3 illustrates typical design specifications of a low-pass filter in step I, where the band \( \{a_0, a_1\} \) (unit 34) is called the passband and \( a_0 \) (unit 36) is the acceptable tolerance (or ripple) in the passband, the band \( \{a_0, a_1\} \) (unit 38) is called the stopband and \( a_0 \) (unit 40) is the corresponding frequency (or ripple), and the band \( \{a_0, a_1\} \) (unit 42) is called the transition band. \( R_s \) is the passband ripple in dB, where \( R_s = -20 \log_{10} \left| \frac{1 - a}{1 + a} \right| \). \( A_s \) is the stopband attenuation in dB, where \( A_s = -20 \log_{10} \left| \frac{a}{1 + a} \right| \). Notably, either \( \{a, a_0\} \) or \( \{R_s, A_s\} \) is required to be stored in step I.
Suppose that an FIR filter has been designed to satisfy the design specifications in step 2. Let \( H(z) = \sum_{n=0}^{\infty} b_n z^{-n} \) be the causal FIR filter with length \( n+1 \). A state-space realization of \( H(z) \) in step 3 can be described as

\[
x(k+1) = Ax(k) + bu(k) \\
y(k) = c^T x(k) + h_0 u(k)
\]

where

\[
A = \begin{bmatrix} 1 & b_1 & \cdots & b_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \\
b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \\ h_0 \end{bmatrix}, \\
c = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix},
\]

and \( A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n, c \in \mathbb{R}^n \). The transfer function \( H(z) \) can also be expressed as \( H(z) = c^T x(z) + h_0 = c^T (zI_n - A)^{-1} b + h_0 \). Our problem formulation is to find a lower-order IIR filter \( F(z) \), which satisfies the same specifications in step 1 as the original FIR filter \( H(z) \) and maintains a linear-phase response in the passband.

The way in the invention is to find an optimal IIR filter by using orthogonal projection of the original FIR filter. By matching some characteristics of the original FIR filter, the resulting orthonormal matrix \( V \) can be generated in step 4. The lower-order IIR filter \( F(z) \) can be constructed using the orthonormal projection \( x(k) = V^T x(k) \). In such a situation, the parameters of the IIR filter can be defined by the following congruence transformation in step 5:

\[
A = V^T A V, \quad b = V^T b, \quad c = V^T c.
\]

It can be shown that the matrix \( V^T A V \) is always stable as long as (1) and matrix \( A \) is stable, (2) \( V^T V = I \). Thus, the stability of the lower-order IIR filter generated by Eq. (3) is guaranteed.

**Pade Approximation and Moment Matching**

The basis theory of the method in the invention is the multi-point Pade approximation, or so called the multi-point moment matching, to obtain a low-order IIR filter. Expanding \( x(z) \) in power series about various frequencies \( \{2_1, 2_2, \ldots, 2_n\} \), where each \( x = x(2)_k \) is obtained, and \( 0 \leq a, b, c \), we have

\[
x(z) = \sum_{j=0}^{\infty} x^{(j)}(2_i) z^{-j},
\]

where

\[
x^{(0)}(2_i) = \left( 2_i - \frac{1}{2} \right) I_n, \\
x^{(1)}(2_i) = -c^T x^{(0)}(2_i)(j > 0), \\
x^{(2)}(2_i) = c^T x^{(1)}(2_i) + h_0.
\]

\( X^{(j)}(2_i) \) is called the \( j \)-th order system moment of \( X(z), \)

\( H^{(j)}(2_i) \) represents the \( j \)-th order output moment of \( H(z) \) at \( 2_i \).

Notably, if \( j=1 \), Eq. (4) is indeed the conventional Pade approximation. The objective is to find a \( q \)-order \( (q+1) \) IIR filter \( F(z) = c^T (zI_n - A)^{-1} b + h_0 \) such that \( H^{(j)}(2_i) = H^{(j)}(2_i) \) for \( j=0, 1, \ldots, n+1 \), where \( q=2, \ldots, n \).

It shall be mentioned that moment calculations can be obtained analytically by exploring special characteristics of matrices \( A \) and \( b \) in Eq. (2). For each \( x \), \( (zI_n - A)^{-1} b \) and \( (zI_n - A)^{-1} \) can be derived analytically as the following formulas:

\[
\begin{bmatrix}
1/\lambda_0 & 0 & \cdots & 0 \\
1/\lambda_1 & 1/\lambda_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1/\lambda_n & 1/\lambda_{n-1} & \cdots & 1/\lambda_0 \\
(2_1 - \lambda_0)^{-1} c_2 & (2_2 - \lambda_0)^{-1} c_3 & \cdots & (2_n - \lambda_0)^{-1} c_1
\end{bmatrix}
\]

Krylov Subspace and the Arnoldi Method

Explicitly computing moments usually yields numerically ill-conditioned problems. We adapt recent results about the Krylov space method to solve these problems. Given a square matrix \( \text{W} \in \mathbb{C}^{m \times n} \) and a vector \( \text{x} \in \mathbb{C}^{n} \), the \( q \)-th Krylov sequence

\[ K_{q}(\text{W}, \text{x}) = \text{x} \times \text{W} \times \text{W}^{2} \times \cdots \times \text{W}^{q-1} \]

is a sequence of \( q \)-vector columns and the corresponding column space is called the \( q \)-th Krylov subspace. Set \( \text{W}_{q} = (zI_n - A)^{-1} \) and \( \text{x}_{q} = (zI_n - A)^{-1} b \). It has been shown that the Krylov subspace \( K_{q}(\text{W}, \text{x}) \) is indeed spanned by the system moments \( x^{(j)}(2_i) \) for \( j=0, 1, \ldots, q-1 \). The Arnoldi method, a kind of Krylov subspace methods, is employed to generate an orthonormal matrix \( V_{q} \) that spans the same subspace as the Krylov subspace \( K_{q}(\text{W}, \text{x}) \). As a result, the guaranteed stable IIR filter can be constructed by substituting \( V_{q} \) into Eq. (3).

The Arnoldi method arises from the Hessenberg reduction \( A = \text{YHV}^{T} \) for eigenvalue calculations. It has the advantage that it can be terminated part-way and leaving one with a partial reduction to a Hessenberg form. The process is exploited to form iterative algorithms. During the iteration process, an upper Hessenberg matrix \( H_{q} \in \mathbb{C}^{m \times n} \) is generated that satisfies the following relationship:

\[ \text{w}_{q} = \text{V}_{q} \times \text{V}_{q}^{T}, \quad \text{x}_{q} = \text{V}_{q} \times \text{V}_{q}^{T}, \quad \text{y}_{q} = \text{V}_{q} \times \text{V}_{q}^{T} \]

where \( q \) is the \( q \)-th unit vector in \( \mathbb{R}^{n} \). The vector \( \text{v}_{q} \) satisfies a \( (q+1) \)-term recurrence relation, involving itself and the preceding Krylov vectors. A new orthonormal vector \( \text{v}_{q+1} \) can be generated using the modified Gram-Schmidt orthonormalization technique.

**The Rational Arnoldi Method**

Generally speaking, the accuracy of the Pade approximation based methods is lost away from the expansion point more rapidly as the eigenvalues of the FIR filter approach the expansion frequency. A rational Arnoldi (RA) method, which uses multiple expansion points, was developed to overcome this difficulty. The straightforward way for multipoint moment matching applications is to apply the Krylov subspace algorithm at various expansion frequencies. This is the so-called rational Krylov algorithm. Basically, this algorithm is a generalization of the shifted-and-inverted Arnoldi algorithm. To simplify the developments, the number of the matched moments of the lower-order IIR filter at each expansion point is assumed to be fixed. Normally, let \( z = \{ z_1, z_2, \ldots, z_L \} \) represent the set of predetermined expansion frequencies. Let \( i = \{ 1, 2, \ldots, n \} \) be the set of the number of the matched moments at each corresponding frequency.
The rational Arnoldi method will generate a lower-order HIR filter $\hat{H}(z)$, which matches $q$-order \( (q=2, \ldots, 1) \) moments of the FIR filter $H(z)$, at the expansion points $z_k = 1, 2, \ldots, f$.

Implementing the rational Arnoldi method is equivalent to applying the Arnoldi method to the expansion point $z_k$. This is the first $j$th iteration. Since no improvement is obtained at the other selected expansion points, the vector $\phi_j(z_k)$ at frequency $x_k$ in the current iteration remains $\phi_{j-1}(z_k)$, which was obtained in the previous iteration. Reset $j = j + 1$ in step 6 and judge if $j = q$ in step 7.

The resulting orthonormal projection matrix $V_j$ is generated in step 8. The resulting orthonormal matrix $V_j$ should be read to achieve that real system matrices of the lower-order IIR filters are generated if the complex expansion frequencies are used.

First, all column vectors in $V_j$ are divided into the real part $V_{jR}$ and the imaginary part $V_{jI}$, Second, a reduced QR factorization of $[V_j, V_j]^T$ is performed to yield a new orthonormal matrix $V_j$. The moment matching property of the resulting lower-order IIR filter by the new and real $V_j$ is also preserved.

The details of the algorithm are outlined as follows. The vector $Z$ includes $q$ expansion points, $q$ is the total number of iterations and $V_j$ is the resulting orthonormal matrix.

### Adaptive Rational Arnoldi (input: $A, b, z, q$; output: $V_j$)

1. **Initialization**
   
   For each $j = 1, 2, \ldots, q$
   
   2. $k^{(p)}(z_k) = (a_k - A)^{-1}b$ for $k^{(0)}(z_k) = k^{(p)}(z_k)$
   
   3. $h_{qj}(z_k) = 1$

   **end for**

2. **Begin the Iterations**

   5. for $j = 1, 2, \ldots, q$

5. **Select the Expansion Frequency with the Maximum Output Moment Error**

6. Choose $z_j = Z_k$ as the $k$th vector that maximizes $\max_{k^{(p)}(z_k)} k^{(p)}(z_k)$

7. set $z_{j+1}$ be the expansion frequency in the $j$th iteration.

8. **Generate the Orthonormal Vector at $z_{j+1}$**

9. $h_{j+1}(z_k) = k^{(j+1)}_k(z_k)$

10. $V_{j+1} = [V_j, h_{j+1}(z_k)]$

11. **Update the Residual $r^{(p)}(z_k)$ for the Next Iteration**

12. for each $z_k$, do

13. if $r_k = 0$, then $k^{(p)}(z_k) = (a_k - A)^{-1} b$

14. else $k^{(p)}(z_k) = k^{(p)}(z_k)$

15. end if

16. $r^{(p)}(z_k) = k^{(p)}(z_k)$

17. for $t = 1, 2, \ldots, f$

18. $h_{j+1}(z_k) = h_{j+1}(z_k) - r^{(p)}(z_k)$

19. end for

20. end for

21. end for

22. $V_j = [V_j, \ldots, V_j]$

Some properties of the method of approximating an FIR filter by lower-order IIR filters in the invention are summarized as follows.

1. Exact expression of output moment errors: Suppose that the output moments of the original FIR filter and those of the lower-order IIR filters are matched, that is, $H(z_k) = H(z_k)$ for $j=0, 1, \ldots, f$, and $j=1, 2, \ldots, f$. The system matrices of the lower-order IIR filter are generated by the congruence transformation with the orthonormal matrix $V_j$ using the algorithm, where \( q = 2, 3, \ldots, f \). The magnitude error between the $j$th-order moments $H(z_k)$ and $H(z_k)$ at each expansion point $z_k$ can be expressed as follows:

\[
\frac{\|H(z_k) - H(z_k)\|}{\|H(z_k)\|} \leq h_{j}(x_k) = \frac{\|H(z_k)\|}{\|H(z_k)\|},
\]

where $h_{j}(x_k) = \frac{\|H(z_k)\|}{\|H(z_k)\|}$. 

### Step 5 in FIG. 4. determines the new residual $r^{(p)}(z_k)$ at each expansion point $z_k$. The calculation involves a projection with the new orthonormal matrix $V_j$. The next vector $k^{(j+1)}(z_k)$ at frequency $x_k$ must be updated to enable further matching of the output moment in the $j+1$th iteration. Since no improvement is obtained at the other selected expansion points, the vector $r^{(p)}(z_k)$ at frequency $x_k$ in the current iteration remains $r^{(p)}(z_k)$, which was obtained in the previous iteration. Reset $j = j + 1$ in step 6 and judge if $j = q$ in step 7, as shown in FIG. 4. Finally, the resulting orthonormal projection matrix $V_j$ is generated in step 8.

The result of the algorithm are outlined as follows. The vector $Z$ includes $q$ expansion points, $q$ is the total number of iterations and $V_j$ is the resulting orthonormal matrix.
(2) Moment matching can still be preserved.

(3) In the first iteration in the rational Arnoldi algorithm with adaptive orders, step (2) is to choose \( z \in \mathbb{C} \) such that
\[
\max \left| e^{i\omega \pi/4} \bar{z} \right| = \max |H(\omega)|.
\]
This is equivalent to find the expansion frequency with the maximum magnitude in the output frequency response.

(4) Implementation issues of digital filters: the present invention also provides several heuristics of selecting expansion frequencies in advance for the proposed rational Arnoldi method. Generally speaking, the complex expansion points \( \{z_1, z_2, \ldots, z_N\} \) will be recommended, where each \( z_i = \omega \Delta \cos \phi \) and \( 0 \leq \phi < 2\pi \). Then the frequency responses of the lower-order IIR filters at these points can be the same as those of the original FIR filter. Nevertheless, if real expansion points can be selected, the computational complexity of yielding approximate FIR filters can be further reduced. The following guidelines are provided:

(a) Low-pass/high-pass filters: the proposed method with the expansion point \( \omega = 0 \) performs well over the low frequency range of responses. For high-pass filter designs, the special structures of state-space matrices may be used to present the duality between low-pass and high-pass filters. Let \( \Lambda = A, B = b, c = c, \) and \( \xi = -h_0, \)
\[ H(z) = z^{-1} (-1)^{-1} (1 + z^2). \]
If \( H(z) \) presents high-pass or low-pass filter, then \( H(z) \) will be a low-pass or high-pass filter, and vice versa. Likewise, if the expansion point \( \omega = 0 \) is chosen to perform the Arnoldi algorithm. If the corresponding orthonormal matrix \( \mathbf{V}_0 \) is obtained, then the high-pass IIR filter, which satisfies the same specifications as the original FIR filter, can be constructed as follows:
\[ \Lambda V_0^T \Lambda, b = V_0^T b, \text{ and } c = V_0^T c. \]
(b) Band-pass/band-stop filters: experimental results indicate that the passband edge and stopband edge frequencies are appropriate candidate expansion points in meeting the specifications of the design. Other expansion points with uniform spacing are also recommended to be selected.

**DESIGN EXAMPLES**

Three example filters are used to justify the proposed approach. Table 1 describes specifications of a low-pass filter, a high-pass filter, and a band-pass filter. The command remez in Matlab was used to design the FIR filters by the optimal equiripple technique. Table 2 lists the corresponding orders. Then, the approximate low-order IIR filters were generated by the proposed method and the balanced realization method (BAL). Table 2 shows the orders and the expansion points used by the two methods. Figs. SA-7C display the bode plots of magnitude, the error in magnitude, and the phase of the original FIR filters and the low-order IIR filters. In Figs. SA-7C, the responses of the original FIR filters are represented as thick solid lines. Those of the IIR filters, determined by the proposed method, are represented as thick dashed lines. The responses in the passband of the IIR filters are indistinguishable from those of the original FIR filters. Independently of which the model reduction method is used. Simulation results imply that the performance of the proposed method is similar to that of the BAL method in the passband. The resulting lower-order IIR filters can actually preserve the linear-phase response of the original FIR filters. Nevertheless, in terms of computational efficiency, the Krylov subspace-based methods generally outperform the BAL method.

**TABLE 1**

<table>
<thead>
<tr>
<th>Specifications</th>
<th>Low-Pass</th>
<th>High-Pass</th>
<th>Band-Pass</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum passband attenuation</td>
<td>( z )</td>
<td>( z )</td>
<td>( z )</td>
</tr>
<tr>
<td>Maximum stopband attenuation</td>
<td>( z )</td>
<td>( z )</td>
<td>( z )</td>
</tr>
<tr>
<td>Lower passband edge (rpm)</td>
<td>0.12</td>
<td>0.12</td>
<td>0.12</td>
</tr>
<tr>
<td>Upper passband edge (rpm)</td>
<td>0.12</td>
<td>0.12</td>
<td>0.12</td>
</tr>
</tbody>
</table>

**TABLE 2**

<table>
<thead>
<tr>
<th>FIR Order</th>
<th>41</th>
<th>41</th>
<th>58</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order</td>
<td>24</td>
<td>17</td>
<td>36</td>
</tr>
<tr>
<td>Band-Pass</td>
<td>27</td>
<td>27</td>
<td>27</td>
</tr>
<tr>
<td>Expansion points</td>
<td>( \omega = 0 )</td>
<td>( \omega = 0 )</td>
<td>( \omega = 0 )</td>
</tr>
</tbody>
</table>

**CONCLUSIONS**

A rational Arnoldi method with adaptive orders for approximating FIR filters by low-order linear-phase IIR filters has been proposed. The developed method is very efficient in terms of computational complexity. Meanwhile, the lower-order IIR filter can truly reflect the dynamical features of the FIR filter and satisfies the original design specifications.

Although the invention has been explained in relation to its preferred embodiment, it is to be understood that many other possible modifications and variations can be made without departing from the spirit and scope of the invention as hereinafter claimed.

What is claimed:

1. A method of approximating an FIR filter with low-order linear-phase IIR filters by the rational Arnoldi algorithm with adaptive orders containing the following steps:
   (a) Initialize the first vector of the Krylov sequence for each expansion point;
   (b) in the jth iteration of the algorithm, choosing an expansion frequency wherein the heuristics of selecting expansion frequencies in advance for the proposed rational Arnoldi method we given by

2. (b) high-pass filters: the special structures of state-space matrices used to present the duality between low-pass and high-pass filters; let state matrices become \( A = A, B = b, c = c, \) and \( \xi = -h_0, \) the expansion point \( \omega = 0 \) chosen to perform the Arnoldi algorithm; and
3. (b) high-pass filters: the special structures of state-space matrices used to present the duality between low-pass and high-pass filters; let state matrices become \( A = A, B = b, c = c, \) and \( \xi = -h_0, \) the expansion point \( \omega = 0 \) chosen to perform the Arnoldi algorithm; and
4. (b) band-pass/band-stop filters: the passband edge and the stopband edge frequencies being the appropriate
candidate expansion points in meeting the specifications of the design, and other expansion points with
uniform spacing recommended to be selected
such that the frequency gives the greatest difference between
the $(j+1)$th-order output moment of the original FIR filter
$H(z)$ and that of the lower-order FIR filter $H(z)$ wherein the
expression of output moment errors between the $j$th-order
moments $H_0(z)$ and $H_0(z)$ at each expansion point $z_j$ are
expressed as follows:

$$|H_0(z_j) - H_0(z_j)| - |w_0e^{j\theta}||$$

where

$$\lambda_0(z_j) = \prod_{j=0}^{n-1} |\psi_i|$$
is the normalization coefficient when an expansion

frequency $z_j$ is selected in the $j$th iteration; vector $c$ contains the

last $n$ impulse response coefficients of a FIR filter with

length $n+1$; and $r^{(j-1)}(z_j)$ is the residual vector in the $(j-1)$th

iteration of the disclosed adaptive rational Arnoldi algorithm

at the expansion frequency $z_j$;

c) after the choosing the expansion point in $j$th iteration

being determined, the single-point Arnoldi method

applied at the expansion point to generate the new

orthonormal vector, and

d) determine a new residual at each expansion point for

next iteration;

whereby, alter the giving total iteration number of the

algorithm, outputting the resulting orthogonal projection

matrix.

* * * * *